The Halting Problem

Lecture 33
Section 12.1

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Outline

Algorithms

Definition

An algorithm is a Turing machine that halts on every possible input.

Decision Problems

Definition

A decision problem is a problem that is clearly stated and unambiguously has a yes-or-no answer.

Decidability

Definition

A decision problem is decidable if there is an algorithm that decides it. Otherwise, the problem is undecidable. That is, a decision problem is decidable if there is a Turing machine that can give the correct yes-or-no answer for every instance of the problem.

Outline

The Halting Problem

Definition (The Halting Problem)

The Halting Problem asks whether there exists a Turing machine H which, when fed as input a representation of a Turing machine M and an input w, will determine whether M would halt if it were executed with input w.

The Halting Problem

Theorem (The Halting Problem)

The Halting Problem is undecidable.

The Halting Problem

Proof.

- Suppose that the Halting Problem is decidable.
- Then there exists a Turing machine H that will decide the Halting Problem.
- We will build a Turing machine M' that does the following.
  - Read a representation of a Turing machine M.
  - Feed M and a copy of M as both machine and input to H to decide whether M would halt if it were executed with M as its own input.
  - If H reports "no," then M' reports "yes."
  - If H reports "yes," then M' loops.

□

Proof.

- Now run M' and feed it a copy of itself as both the Turing machine and its input.
- M' must halt or loop.
- If M' reports "yes," then H must have reported "no," meaning that M' does not halt.
- But M' did halt, so that is a contradiction.
- If M' loops, then it is because H reported "yes," meaning that M' halts.
- But M' did not halt, so that, too, is a contradiction.
- The conclusion is that no such Turing maching H exists.

$\square$

Outline

Reduction of Problems

Definition (Reduction of Problems)

Given two decision problems A and B, a reduction of A to B is an algorithm (Turing machine) that will convert any instance of A into an instance of B such that the two instances have the same yes-or-no answer.

The Membership Problem

Definition

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- We can reduce the membership problem to the halting problem.
- Can we reduce the halting problem to the membership problem?

Reducing the Halting Problem

- Let A be a decision problem and let H be the halting problem.
- If H can be reduced to A, then A is undecidable.

The State-Entry Problem

Definition

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- The halting problem can be reduced to the state-entry problem.
- Therefore, the state-entry problem is undecidable.

Outline

Assignment

Homework

- Section 12.1 Exercises 2, 3, 7, 8, 12, 13.